FACULTY WORKING PAPER 91-0132

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BEBR

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April 1991

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We are grateful to Rick Vinod, Vicki Zinde-Walsh and Mann Yoon for comments on an earlier draft of the paper. Financial support from the Research Board and the Bureau of Economics and Business Research of the University of Illinois is gratefully acknowledged.

ABSTRACT

Rao's work is always inspired by some practical problems. It is not surprising that many of the techniques he developed found their way to various applications in econometrics. In this review paper, we try to evaluate the impact of his pioneering 1948 score test on the current practice of econometric model specification tests. In so doing, we relate Rao's score principle to some of the earlier popular tests in econometrics. We also discuss the recent developments on this topic and problems for further research.

1. INTRODUCTION

The history of statistical hypothesis testing is indeed a very long one. However, if one has to pick a few papers that have profound influence on the current practice, the following three papers must be among those: Neyman and Pearson (1928), Wald (1943) and Rao (1948). These papers suggested three basic principles of testing, likelihood ratio (LR), Wald (W) and Rao's score (RS), respectively. For a long time econometricians have been invariably using the LR test and some version of the Wald principle. Score test was rarely used explicitly, although some of the earlier econometric tests could be given score test interpretation. In the late seventies and eighties, however, we observed intense activities in the application of score principle to a variety of econometric testing problems and studying the properties of the resulting tests. And now RS tests are the most common items in the econometricians' kit of testing tools.

The aim of this review paper is very modest. Our main purpose is to highlight the role of RS test in the current development of model specification and diagnostic checks in econometrics. Applications of RS test in econometrics are so vast and the number of papers on this topic is so numerous that it is an impossible task to provide a balanced and comprehensive review. There are already some survey papers and monographs that cover RS test. In particular, we would refer the readers to Breusch and Pagan (1980), Engle (1984), Kramer and Sonnberger (1986) and Godfrey (1988). Most of the recent textbooks also discuss RS test, for example, see White (1984, pp. 72-74), Amemiya (1985, pp. 141-146), Judge et al. (1985, pp. 182-187), Kmenta (1986,

pp. 493-495), Spanos (1986, pp. 326-336), Maddala (1988, pp. 137-139), Green (1990, pp. 357-359) and Harvey (1990, pp. 169-177).

The plan of the paper is as follows. In the next section, we define the RS test statistic and discuss some of its properties, particularly its invariance and asymptotically equivalent forms. In Section 3, we mention some of the popular applications in econometric model evaluation. Many of the old and new econometric tests could be given score test interpretation, and these are mentioned in Section 4. There are several advantages of RS test compared to the LR and W tests and Section 5 discusses that aspect. There still remain many unsolved problems. In Section 6, we provide a brief review of recent developments and mention some problems for further research. Last section concludes the paper with a few remarks.

2. SCORE TEST AND ITS PROPERTIES

Suppose there are n independent observations, $y_1, y_2, \ldots, y_n$ with identical density function $f(y;\theta)$ where θ is a $p \times 1$ parameter vector with $\theta \epsilon \Theta \subseteq R^p$. It is assumed that $f(y;\theta)$ satisfies the regularity conditions stated in Rao (1973, p. 364) and Serfling (1980, p. 144). The log-likelihood function, the score vector and the information matrix are then defined, respectively, as

$$\ell(\theta) = \sum_{i=1}^{n} \ell n\, f(y_i, \theta)$$

$$d(\theta) = \frac{\partial \ell(\theta)}{\partial \theta}$$

and

$$I(\theta) = -E\left[\frac{\partial^2 \ell(\theta)}{\partial\theta\partial\theta'}\right]$$

The hypothesis to be tested is H_o: $h(\theta) = c$ where $h(\theta)$ is an $r \times 1$ vector function of θ with $r \leq p$ and c is a known constant vector. It is assumed that $H \equiv H(\theta) = \partial h(\theta)/\partial\theta$ has full column rank, i.e., rank $[H(\theta)] = r$. The RS statistic for testing H_o can be written as

$$RS = d'(\tilde{\theta}) I(\tilde{\theta})^{-1} d(\tilde{\theta}) \qquad (2.1)$$

where "~" indicates that the quantities have been evaluated at the restricted maximum likelihood estimate (MLE) of θ, say $\tilde{\theta}$.

The idea behind this test is that if H_o is true, $d(\tilde{\theta})$ is expected to be close to zero by virtue of the fact that $d(\hat{\theta}) = 0$, where $\hat{\theta}$ is the unrestricted MLE of θ. Under H_o, RS is asymptotically distributed as χ^2_r [see, for example, Rao (1973, pp. 418-419), Serfling (1980, pp. 156-160) and Godfrey (1988, pp. 13-15)]. Econometricians use the above score form of the test. However, it is often called as the Lagrange multiplier (LM) test. The terminology LM test came from the two articles Aitchison and Silvey (1958) and Silvey (1959), where an LM interpretation of (2.1) was given.

Note that $\tilde{\theta}$ can be obtained from the solution to the equations

$$d(\tilde{\theta}) - H(\tilde{\theta})\tilde{\lambda} = 0$$

and

$$h(\tilde{\theta}) = c.$$

where λ's are the Lagrange multipliers. Therefore, we have $d(\tilde{\theta}) = H(\tilde{\theta})\tilde{\lambda}$. Given that $H(\tilde{\theta})$ has full rank, $d(\tilde{\theta}) = 0$ is equivalent to $\tilde{\lambda} = 0$, i.e., the Lagrange multipliers vanish. These multipliers can be interpreted as the implicit cost (shadow prices) of imposing the restrictions. It can be shown that

$$\tilde{\lambda} = \frac{\partial \ell(\tilde{\theta})}{\partial c}$$

that is, the multipliers give the rate of change of the maximum attainable value with respect to the change in the constraints. If $H_o : h(\theta) = c$ is true and $\ell(\tilde{\theta})$ gives the optimal value, $\tilde{\lambda}$ should be close to zero. Given this "economic" interpretation in terms of multipliers, it is not surprising that the econometricians prefer the term LM rather than RS. In terms of Lagrange multipliers, (2.1) can be expressed as

$$LM = \tilde{\lambda}' H'(\tilde{\theta}) I(\tilde{\theta})^{-1} H(\tilde{\theta}) \tilde{\lambda} \qquad (2.2)$$

It is clear from (2.1) and (2.2) that RS form of the test is much easier to compute, and this explains its popularity among econometricians. However, as explained above, they call it by a name which is closer to economists' way of thinking.

From (2.1) it is clear that the RS statistic is essentially a distance measure between the null and alternative hypotheses, and it can be given Mahalanobis norm interpretation [see Del Pino (1990)]. This leads to the development of alternative forms of RS test by using

different types of distance measures [see Ullah (1989)]. Note that the RS statistic does not depend on the alternative hypothesis explicitly, since it uses the slope of $\ell(\theta)$ at $\theta = \tilde{\theta}$. There may be different likelihood function (having the same slope and possibly the same curvature at H_0) that will lead to the same RS statistic. This phenomenon is called the invariance property of the RS test. There are many examples of this, but here we mention only a few. The RS statistic for testing normality suggested in Bera and Jarque (1981) and Jarque and Bera (1987) with Pearson family of distributions as the alternative, remains unchanged under Gram-Charlier (type A) alternatives [see Bera (1982a, p. 98)]. Statistics for testing homoskedasticity are invariant to different forms of alternatives such as multiplicative and additive heteroskedasticities as has been noted by Breusch and Pagan (1979) and Godfrey and Wickens (1982). Testing serial independence against the p-th order autoregressive [AR(p)] or the p-th order moving average [MA(p)] processes lead to the same test statistic, see for instance Breusch (1978) and Godfrey (1978a). Pesaran (1979) found that the RS test does not differentiate between polynomial and rational distributed lags.

These examples raise the question whether the RS test will be inferior to other asymptotically equivalent procedures such as the W and LR tests, with respect to power, since it does not use precise information of the alternative hypothesis. However, the Monte Carlo results of Godfrey (1981) and Bera and McKenzie (1986) suggest that apparently there is no setback in the performance of the RS test compared to the LR test. Using locally equivalent alternative (LEA)

models, Godfrey and Wickens (1982) provide some theoretical underpinnings behind the invariance property of the RS test. Their analyses also highlight the fact that a proper choice of the alternative sometimes greatly reduces the computational complexity of a test. The invariance property also cautions researchers against automatically accepting the alternative hypothesis when the null is rejected. That is, if we reject the null, it does not imply that the alternative that has been used to derive the test is correct.

We can calculate the variance of $d(\theta)$ in a number of ways which are asymptotically the same. This leads to different versions of the RS test with different properties in small samples. In fact, in the general formula (2.1), if we substitute any positive definite matrix $A(\theta)$ in place of $I(\theta)$ such that

$$plim\left[A(\tilde{\theta})^{-1}\left(\frac{\partial^2 \ell(\tilde{\theta})}{\partial\theta\partial\theta'}\right)\right] = I_p$$

where I_p is a $p \times p$ identity matrix, then all the statistics of the form $d'(\tilde{\theta})A(\tilde{\theta})^{-1}d(\tilde{\theta})$ will be asymptotically equivalent. Two immediate choices of $A(\theta)$ are

$$A_1(\theta) = E[d(\theta) \cdot d'(\theta)] \tag{2.3a}$$

and

$$A_2(\theta) = E\left[-\frac{\partial^2 \ell(\theta)}{\partial\theta\partial\theta'}\right] \tag{2.3b}$$

If the specified probability model is correct then $A_1(\theta) = A_2(\theta)$ [see White (1982)]. In that case either $A_1(\theta)$ or $A_2(\theta)$ could be used, and we will denote the statistic as RS(WE), "WE" stands for "with expectation" using the terminology in Bera and McKenzie (1986).

In certain situations, it may be difficult to calculate the above expectations analytically, for example, when testing linear and log-linear models using Godfrey and Wickens (1981) approach. In such cases we can use the Hessian form

$$A_3(\theta) = -\frac{\partial^2 \ell(\theta)}{\partial\theta\partial\theta'} \tag{2.4a}$$

or the outer product form

$$A_4(\theta) = G(\theta)G'(\theta) \tag{2.4b}$$

where $G(\theta)$ has typical (i,j)-th element as $\partial\ell_i(\theta)/\partial\theta_j$, $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,p)$ with $\ell_i = \ln f(y_i,\theta)$. When $A_4(\theta)$ is used, noting that $d(\tilde{\theta}) = G'(\tilde{\theta})\underline{1}$, the RS statistic can be computed as

$$\begin{aligned} RS &= \underline{1}'G(\tilde{\theta})[G'(\tilde{\theta})G(\tilde{\theta})]^{-1}G'(\tilde{\theta})\underline{1} \\ &= nR^2 \end{aligned} \tag{2.5}$$

where $\underline{1}$ is an $n \times 1$ vector of ones and R^2 is the uncentered coefficient of determination of regressing $\underline{1}$ on $G(\tilde{\theta})$ [see Godfrey and Wickens (1981)]. Bera (1982b) argued that this statistic can be written in Hotelling's T^2 form, and for finite samples it is approximately distributed as an $nr/(n - r + 1)$ multiple of $F(r, n - r + 1)$ under H_o.

For future reference, we will denote this form as RS(WOE) to mean "without expectation." We note here that A_1, A_2, A_3 and A_4 are based on Shanon's entropy. Other choices of A's, and hence the variants of RS test can be developed by using various measures of entropy [see Ullah (1989)].

Davidson and MacKinnon (1984a) have proposed an alternative version of RS test that may improve the small sample properties. It can also be obtained by running a regression. As their regression is based on 2n observations, it is called RS(DLR) to emphasize the nature of double length regression.

All the above versions are based on the assumption that the underlying probability model is correctly specified. When this assumption fails, the above versions of RS test will not have correct size even asymptotically. To overcome this problem White (1982) proposed using [see also Kent (1982)]

$$A_5(\theta) = A_3(\theta)^{-1} A_4(\theta) A_3(\theta)^{-1} \tag{2.6}$$

White's version can also be expressed in nR^2 form, where R^2 is the uncentered coefficient of determination of the regression of a unit vector on $G(\tilde{\theta})A_3(\tilde{\theta})^{-1}H(\tilde{\theta})$ [see Bera and McKenzie (1986)].

If the maintained hypothesis is correct all the above forms are asymptotically equivalent. The asymptotic equivalence of the tests is not necessarily indicative of their finite sample behavior. Using $A_3(\theta)$ sometimes might lead to negative values of the test statistic in small sample and also this version will not be invariant to units of

measurement. Moreover, $I(\tilde{\theta})$ contains fewer stochastic terms than $A_3(\theta)$ and therefore, RS(WE) can be expected to converge to χ^2_r faster than RS(WOE) under H_o. On the other hand Efron and Hinkley (1978) argued that $A_3(\theta)$ and $A_4(\theta)$ are "closer to the data" than $I(\theta)$ and should be favored. Davidson and MacKinnon (1983) and Bera and McKenzie (1986) examined the small sample performances of various versions of RS statistic. Their recommendations were to use RS(WE) or RS(DLR). In certain cases RS(WE) is difficult to obtain whereas RS(DLR) can be easily calculated. An example is the problem of testing linear model within the Box-Cox transformation framework. There are also circumstances in which RS(DLR) is not applicable such as testing for misspecification in certain limited dependent variable models. When both RS(WE) and RS(DLR) are available, either of them can be used since there is not much difference between these two versions of RS statistic in terms of small sample size and power.

One of the most interesting properties of the RS test is its additivity in the sense that the RS statistic for testing two (or more) hypotheses jointly can be decomposed into sum of the RS statistics for testing the hypotheses individually. This property was first noted by Pesaran (1979). He found that the RS test of the deterministic and stochastic parts of a dynamic linear regression model can be written as the sum of two independent parts. In a more complex situation, Bera and Jarque (1982) showed that the joint RS test for normality, homoskedasticity, serial independence and functional form is the sum of the standard RS tests for each component of the joint hypothesis. More recently, Higgins and Bera (1989) suggested a simultaneous test for

autoregressive conditional heteroskedasticity (ARCH) and bilinearity which is just the sum of individual tests for ARCH and bilinearity.

Let us assume that $H_0 : h(\theta) = c$ naturally partitions into two separate sets of restrictions, $H_A : h_1(\theta) = c_1$ and $H_B : h_2(\theta) = c_2$ with a corresponding partition for $H(\theta) = \partial h(\theta)/\partial\theta$ as $H = [H_1 : H_2]$. For a test principle T, denote by T_{AB} the statistic for simultaneously testing both sets of restrictions H_A and H_B. Let T_A denote the statistic for testing H_A with the H_B restrictions imposed, and similarly T_B is defined. With respect to the test principle T, the tests for hypotheses H_A and H_B are said to be additive if $T_{AB} = T_A + T_B$. Bera and McKenzie (1987) showed that a necessary and sufficient condition for additivity of the RS tests for the hypotheses H_A and H_B is that $H_1'(\tilde{\theta})\mathcal{I}(\tilde{\theta})^{-1}H_2(\tilde{\theta}) = 0$.

Aitchison (1962) introduced the concept of separability; if two hypotheses are separable and the sample size is large, while testing one hypothesis we may be able to ignore the other hypothesis. That is, the test is "robust" to whether the other hypothesis is true or not. This idea is very much related to Stein's (1956) "adaptive" test. Results of Bera and McKenzie (1987) imply that separability of the RS test also leads to its additivity. However, for the LR test this is true only asymptotically and for the W test the result does not hold in general. After carrying out one-directional RS tests, when additivity applies, a joint test can be obtained by simply adding up the component statistics. This provides an optimal way to combine different diagnostic tests of econometric models [see Pagan and Hall (1983, p. 152) and Pagan (1984,

p. 126)]. For the LR and sometimes for W tests, such an operation is valid only asymptotically.

3. SOME SPECIFIC APPLICATIONS OF RS TESTS TO ECONOMETRIC MODEL EVALUATION

Byron (1968) was probably the first to introduce the RS test to the econometrics literature. He used Silvey's LM version for testing linear restrictions in demand system. However, we had to wait more than a decade to realize the potential of the RS test in econometrics. The earlier notable papers were Savin (1976), Berndt and Savin (1977), Breusch (1978, 1979) and Godfrey (1978a,b,c). Possibly, Breusch and Pagan (1980) has been the most influential. It presented the RS test in a general framework in the context of econometric model evaluation and discussed many applications. Engle's (1984) survey provided excellent progress report up to early eighties. Now there is a full length monograph, Godfrey (1988) which gives a very comprehensive account of most of the available RS tests in econometrics. And there is no doubt that in coming years a few more survey papers and monographs will be written. In this section, we do not make any attempt to provide a complete list of all applications of the RS test in econometrics. We just mention some of the commonly used RS tests and discuss their links with the older tests.

One of the oldest test in statistics and econometrics is the Pearson χ^2 goodness-of-fit test. Using a multinomial likelihood function it can be easily shown that this is a RS test [see Rao (1973, p. 442)]. The analysis of classical linear regression model

$$y_i = x_i'\beta + \varepsilon_i, \quad i = 1,2,\ldots,n \tag{3.1}$$

is often based on four basic assumptions, correct linear functional form and the assumptions of disturbance normality, homoskedasticity and serial independence. Violation of these affects both the estimation and inference results. Until recently these basic assumptions were not tested thoroughly. With the aid of RS principle, numerous procedures have been proposed to test the above assumptions and these are now routinely reported in most of the standard econometric packages such as PC-GIVE, MICROFIT and SHAZAM. Just to name some of the RS test procedures, test for normality has been derived by Bera and Jarque (1981) and Jarque and Bera (1987), Godfrey (1978c) and Breusch and Pagan (1979) proposed tests for homoskedasticity and Breusch (1978) and Godfrey (1978a,b) developed tests for serial independence. The older tests like Durbin and Watson (1950), Chow (1960), Goldfeld and Quandt (1965), Durbin (1970) could also be given RS test interpretation.

To see the attractiveness of the RS test, let us briefly consider the test for normality. Bera and Jarque (1981) started with the Pearson family of distributions. Given the complexity of the ML estimation of the Pearson family, W and LR tests are ruled out from a practical point of view. However, the RS test is based on

$$RS = n\left[\frac{(\sqrt{b_1})^2}{6} + \frac{(b_2 - 3)^2}{24}\right] \tag{3.2}$$

where $\sqrt{b_1} = \mu_3/\mu_2^{3/2}$, $b_2 = \mu_4/\mu_2^2$ and $\mu_j = \sum_{i=1}^{n} \tilde{\varepsilon}_i^j/n$ where $\tilde{\varepsilon}_i = y_i - x_i'\tilde{\beta}$ are the OLS residuals. It turns out that this test was also mentioned by

Bowman and Shenton (1975) and was hardly used in statistical work due to lack of theoretical justification. Now we note that RS approach uncovers a principle that proves the asymptotic efficiency of the test in (3.2). As we mentioned earlier if Gram-Charlier (Type-A) expansion is used instead of the Pearson family as the alternative distribution, the same test would result. Furthermore, once the test statistic or just the score vector is derived, for the RS test, the alternative hypothesis is irrelevant. The above test is based on two moments, third and fourth. One could have started with these moments directly without going through the derivation. In that sense RS test has moment test interpretation [see Newey (1985), Tauchen (1985) and Pagan and Vella (1989)].

As mentioned previously, the first application of RS test to econometrics was in testing linear restriction in the demand system [Byron (1968)]. The null hypothesis can be stated as $H_o : R\beta = c$ for the model (3.1) where R is a known constant matrix. Berndt and Savin (1977) showed for this problem, under the assumption of normality of ε, $W \geq LR \geq RS$ [see also Rothenberg (1984)]. This implies that if we use the same critical value for the three statistics, it might result in conflicting inference. Geweke (1981) reached similar conclusion by comparing the Bahadur slopes of these three tests for linear restriction in the regression model with nonscalar covariance matrix [also see Magee (1987)]. When $V(\varepsilon) = I_n\sigma^2$, W is a linear function of the F-test and exact critical values for all the three tests can be obtained [see Bera (1982c)]. Ullah and Zinde-Walsh (1984) studied the properties of the three test statistics particularly the above inequalities when the

disturbance term follows a multivariate t-distribution. They found that although there was no change in the LR statistic, the inequalities were no longer true. The relative magnitude of the degree of freedom of the t-distribution with respect to the sample size was the crucial factor. Ullah and Zinde-Walsh (1985) and Zinde-Walsh and Ullah (1987) generalized these results to spherically symmetric and elliptical error distributions, respectively.

Apart from its local asymptotic efficiency, one of the major attractions of the RS test, as we stressed before, is its computational simplicity. However, with the current stage of computing, this of course is not a decisive factor. What is attractive is that in most cases we can represent the RS test simply as a variable addition test in the context of regression model. To demonstrate this through a simple case, let us consider a simplified version of the Durbin (1970) example. Suppose we have the following regression model

$$y_i = \alpha y_{i-1} + x_i'\beta + u_i, \quad i = 1,2,\ldots,n$$

$$u_i = \theta_2 u_{i-1} + \varepsilon_i, \qquad |\theta_2|<|$$

$$\varepsilon_i \sim IIDN(0,\sigma^2)$$

and we are interested in testing $H_o : \theta_2 = 0$. There are a number of ways to carry out the RS tests. One simple way is to test significance of the coefficient of $\tilde{u}_{i-1}$ in the following artificial regression

$$y_i = \alpha y_{i-1} + x_i'\beta + \theta_2 \tilde{u}_{i-1} + \omega_i \tag{3.3}$$

where $\tilde{u}_i$'s are the OLS residuals [see Godfrey and Wickens (1982)]. It is clear from (3.3) that the test is based on $\tilde{\theta}_2 = (\sum_i \tilde{u}_i\tilde{u}_{i-1}/\sum_i \tilde{u}_{i-1}^2)$, an ML type estimate of θ_2. There are many instances where RS tests can be performed by adding certain extra variables to the original regression model and examining the significance of these extra regressors. For more on variable addition tests and artificial regression approach to testing, see Pagan (1984) and MacKinnon (1990).

Tests developed for the standard regression model have been generalized to testing for the limited dependent variable, discrete choice, disequilibrium and simultaneous equation models. Specification tests are more important for these models since violation of ideal conditions have more serious consequences for these specialized econometric models. A large number of papers discuss the details of RS test for these models [for example, see Jarque and Bera (1982), Bera, Jarque and Lee (1984), Davidson and MacKinnon (1984b), Engle (1984), Lee and Maddala (1985), Robinson, Bera and Jarque (1985), Cameron and Trivedi (1986), Moon (1988), Bera and Robinson (1989) and Pagan and Vella (1989)]. Recently, Pagan and Ullah (1990, ch. 6) extended the variable addition approach to RS tests for the case where the linear part in (3.1) is replaced by a nonparametric regression. Finally, using the RS principle, Anselin (1988a,b) developed several diagnostics for assessing model misspecification due to spatial dependence and heterogeneity.

4. SCORE INTERPRETATION OF SOME ECONOMETRIC TESTS

One of the common problems in econometrics involves testing two economic models which are non-nested, in the sense that one is not a special case of other, or one cannot be obtained from the other under some limiting values of the parameters. For this problem standard LR test is not applicable. Cox (1961, 1962) suggested a test which is based on the ratio of maximized loglikelihoods less an estimate of its expected value under the tested hypothesis. Breusch and Pagan (1980, p. 248) made the first attempt to relate the Cox approach to RS test. However, there was some confusion as indicated in Pesaran (1982), due to the fact that under the tested hypothesis parameters of the alternative model are not identified within the comprehensive model framework. This is the familiar Davies (1977, 1987) problem. Dastoor (1985) clearly showed that Cox test is essentially a RS test if the parameters of the alternative hypothesis are replaced by their estimates before applying the score principle.

During the sixties when econometrics was rather in a wilderness stage, Durbin (1970) solved some of the puzzles in model testing, particularly the use of Durbin-Watson test in the presence of lagged dependent variables. Suppose we partition θ as $\theta = [\theta_1' : \theta_2']'$ and want to test $H_o : \theta_2 = \theta_{2o}$.

Let us define

$$\tilde{\theta}_1 = MLE\ of\ \theta_1\ given\ \theta_2 = \theta_{2o}$$

and

$$\tilde{\theta}_2 = \textit{MLE of } \theta_2 \textit{ given } \theta_1 = \tilde{\theta}_1$$

Durbin's (1970) test is based on $(\tilde{\theta}_2 - \theta_{2o})$. However, we can write

$$0 = \frac{\partial \ell(\tilde{\theta}_1, \tilde{\theta}_2)}{\partial \theta_2} \stackrel{a}{=} \frac{\partial \ell(\tilde{\theta}_1, \theta_{2o})}{\partial \theta_2} + \frac{\partial^2 \ell(\tilde{\theta}_1, \theta_2^*)}{\partial \theta_2 \partial \theta_2'} (\tilde{\theta}_2 - \theta_{2o})$$

where θ_2^* lies in between $\tilde{\theta}_2$ and θ_{2o}. Since in large sample the Hessian matrix can be taken to be nonsingular, a test based on $(\tilde{\theta}_2 - \theta_{2o})$ is asymptotically equivalent to a test that uses $\partial \ell(\tilde{\theta}_1, \theta_{2o})/\partial \theta_2$. This latter quantity is the score vector with respect to θ_2 and the RS test for $H_o : \theta_2 = \theta_{2o}$ uses exactly this.

Hausman (1978) test is based on a simple principle [see also Durbin (1954)]. Assume that under the null hypothesis of no misspecification, the consistent, asymptotically normal and efficient estimator of θ is $\tilde{\theta}$, which under the alternative hypothesis of misspecification is inconsistent. Suppose there is another estimator, $\hat{\theta}$ which is consistent (with the same rate of convergence as $\tilde{\theta}$) under the null as well as under the alternative. Then the Hausman test considers the quantity $\omega = \hat{\theta} - \tilde{\theta}$. [It is also interesting to note that Hausman's computation of the asymptotic variance of $\sqrt{n}\omega$ as $V(\sqrt{n}\hat{\theta}) - V(\sqrt{n}\tilde{\theta})$ under the null is based on a theorem by Rao (1973, p. 317)]. Let us now consider a special case of testing $H_o : \theta_2 = \theta_{2o}$ where $\theta = (\theta_1', \theta_2')'$. We use the same notations as before. In particular, $\tilde{\theta}_1$ is the MLE of θ_1 given $\theta_2 = \theta_{2o}$ and $\hat{\theta} = (\hat{\theta}_1', \hat{\theta}_2')'$ is the unrestricted MLE. As interpreted by Holly (1982), Hausman's test is based on comparing the two estimates

$\tilde{\theta}_1$ and $\hat{\theta}_1$. Since $\tilde{\theta}_1$ is efficient under H_o but usually inconsistent under H_1, while $\hat{\theta}_1$ is consistent under H_o as well as H_1, they are obvious candidates to use in constructing the Hausman test, which is based on $\hat{\theta}_1 - \tilde{\theta}_1$. It can be shown that [see Ruud (1984) and Godfrey (1988, p. 34)]

$$\frac{\partial \ell(\hat{\theta}_1, \theta_{2o})}{\partial \theta_1} \underset{=}{a} \frac{\partial^2 \ell(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_1'} (\hat{\theta}_1 - \tilde{\theta}_1) \tag{4.1}$$

Therefore, the Hausman test has score interpretation. However, note that in (4.1) the score is with respect to the nuisance parameter θ_1 and it is evaluated at $\hat{\theta}_1$ not $\tilde{\theta}_1$. Here $(\hat{\theta}_1, \theta_{2o})$ can be viewed as consistent estimator for θ under the null hypothesis. When consistent (rather than efficient) estimators are used there is another attractive way to construct a score type test which is due to Neyman (1959, 1979). In the literature this is known as $C(\alpha)$ or effective score or Neyman-Rao test [see Hall and Mathiason (1990)]. For this test we need $\sqrt{n}$-consistent estimator. To avoid confusion let us denote $\sqrt{n}$-consistent estimator of θ under $H_o : \theta_2 = \theta_{2o}$ as $\theta^+ = (\theta_1^{+\prime}, \theta_{2o}')'$. Then Neyman-Rao $C(\alpha)$ test uses the effective score

$$\frac{\partial^* \ell(\theta)}{\partial \theta_2} = \frac{\partial \ell(\theta^+)}{\partial \theta_2} - I_{21}(\theta^+) I_{11}^{-1}(\theta^+) \frac{\partial \ell(\theta^+)}{\partial \theta_1} \tag{4.2}$$

where I_{ij} are the appropriate blocks of the information matrix $I(\theta)$ corresponding to θ_1 and θ_2. It is clear that when $\tilde{\theta}$ is used in place of θ^+ in (4.2), $\partial \ell(\theta)/\partial \theta_1 = 0$ and the test reduces to the standard RS test.

In a sense, $\partial^*\ell(\theta)/\partial\theta_2$ is the residual score obtained from the residual of running the regression of $\partial\ell_i(\theta)/\partial\theta_2$ on $\partial\ell_i(\theta)/\partial\theta_1$. Therefore, $\partial^*\ell(\theta)/\partial\theta_2$ is the part of score which is orthogonal to the score for θ_1 [see Neyman (1979) and Hall and Mathiason (1990)]. From a practical point of view NR test is very attractive, firstly all we need is $\sqrt{n}$-consistent estimator and secondly the test can be viewed as adaptive in the sense of Stein (1956) because the test is independent of the value of nuisance parameter θ_1. This Neyman-Rao orthogonalization procedure has wider implications. Recently Bera and Yoon (1990) have applied this approach to develop tests which are valid under misspecified alternatives.

In Section 2, we noted that when the model is correctly specified then $A_1(\theta) = A_2(\theta)$ [see equations in (2.3)]. This equality can be used to construct a test for model specification. White (1982) suggested a specification test which examines whether the elements of $A_1(\hat{\theta}) - A_2(\hat{\theta})$ are close to zero where $\hat{\theta}$ now denotes quasi-MLE. This test is known as information matrix (IM) test since it exploits the IM equality. Consider a vector $s(y;\theta)$ defined by

$$s(y;\theta) = vech\left[\frac{\partial^2 lnf(y;\theta)}{\partial\theta\partial\theta'} + \frac{\partial lnf(y;\theta)}{\partial\theta}\,\frac{\partial lnf(y;\theta)}{\partial\theta'}\right]$$

where "vech" denotes an operator that stacks distinct elements of a symmetric matrix. Thus $s(y;\theta)$ is a vector with $p(p+1)/2$ elements, generally referred to as indicators. The actual test utilizes a sample average of $s(y,\theta)$, namely

$$\bar{s}(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n} s(y_i,\hat{\theta})$$

If the model is correctly specified $\bar{s}(\hat{\theta})$ should take small values. IM test has found many applications in econometrics, for example, see Hall (1987) and Bera and Lee (1990). Chesher (1984) provided a very interesting interpretation of the IM test. He considered the problem of testing for random parameters and assumed that θ has mean say $\bar{\theta}$ and variance Γ, a $p \times p$ matrix. The null hypothesis that θ is nonstochastic is equivalent to $H_0 : \Gamma = 0$. For Γ close to zero, the marginal density of y can be approximated by

$$f^{*}(y;\bar{\theta},\Gamma) = f(y;\bar{\theta})\left[1 + tr\{(F_2 + F_1F_1')\Gamma\}\right]$$

where $F_1 = \partial \ln f/\partial\theta$ and $F_2 = \partial^2 \ln f/\partial\theta\partial\theta'$. It is clear that under $H_0 : \Gamma = 0$, $f^*(y;\theta,0) = f(y;\theta)$. Chesher showed that the score vector for testing H_0 is precisely equal to $n\bar{s}(\hat{\theta})$. Thus the IM test has a score test interpretation.

5. ADVANTAGES OF THE RS TEST

Apart from its computational simplicity and local asymptotic efficiency, RS test has advantages from some other theoretical and practical points of view. For example, consider the case of testing when under the null hypothesis the parameter values lie on the boundary of the parameter space. In this context, the standard theory associated with tests based on MLE's will not be valid. Large sample properties of MLE's and the associated tests in boundary situations have been examined

by Chernoff (1954), Moran (1971), Chant (1974) and Self and Liang (1987). One important general result from their investigation is that the W and LR tests in the boundary situation will not follow their usual asymptotic χ^2-distribution while the asymptotic properties of the RS test are not altered. As a result it has been argued that the RS test is particularly suitable for testing hypotheses under which parameter values are at the boundary of the parameter space [see Godfrey (1988, p. 95)].

Sometimes the W test might have some computational advantages. For models in which the null hypothesis imposes nonlinear restrictions on θ, the unrestricted MLE $\hat{\theta}$ is easier to calculate. Examples of such cases can be found in the dynamic specification test of Sargan (1980), the rational expectation hypothesis test of Wallis (1980) and Hoffman and Schmidt (1984) and the test of market efficiency under rational expectation as described in Baillie, Lippens and McMahon (1983). Although the W test might be easier, particularly in this kind of situation it runs into a serious problem. Gregory and Veal (1985) pointed out that the numerical value of the W statistic is not invariant to the algebraically equivalent forms of the null hypothesis. However, the RS test is invariant to different equivalent forms of nonlinear restrictions. The main problem with the W test is that it uses a wrong "metric" and is not invariant to changes in co-ordinates [see Critchley, Marriott and Salmon (1990) and Davidson (1990)].

Finally, let us consider the problem of testing $H_o : \theta_2 = \theta_{2o}$ when the density function depends on $\theta = (\theta_1', \theta_2')'$. Suppose given a value of θ_1, we can find a test statistic for testing H_o. In many econometric

problems it happens that when H_o is true the model is free of θ_1. In other words, the nuisance parameter θ_1 is identified only under the alternative hypothesis. This implies under H_o, the information matrix is singular thus invalidating the standard test procedures. However, using the results of Davies (1977, 1987), it is easy to modify the RS test to obtain a valid test procedure. Suppose given θ_1, RS statistic for testing H_o is given by $RS(\theta_1)$. Davies approach appeals to the "union-intersection principle" of Roy (1953) and suggest basing the test on a critical region of the form

$$\left\{ \sup_{\theta_1} RS(\theta_1) > k \right\}$$

where k is chosen to have an appropriate size of the test. In practice this method works very well as has been demonstrated by Bera and Higgins (1990).

6. SOME RECENT DEVELOPMENTS

To provide an intuitive idea behind the score test, earlier we noted that if H_o is true, $d(\tilde{\theta})$ should be close to zero since by construction $d(\hat{\theta}) = 0$. However, Conniffe (1987, 1988) argued that a better justification comes from the fact that $E[d(\theta)] = 0$ at the true value of θ [see also Conniffe (1990a)]. When H_o is a simple hypothesis, then there is no problem. However, for testing a composite hypothesis, we need to replace some components of θ by their estimates in $d(\theta)$. Then $E[d(\tilde{\theta})]$ may no longer be zero. In that case Conniffe suggested using

$$[d(\tilde{\theta}) - E\{d(\tilde{\theta})\}]'J(\tilde{\theta})[d(\tilde{\theta}) - E\{d(\tilde{\theta})\}]$$

where J is the inverse of the variance of $d(\tilde{\theta}) - E[d(\tilde{\theta})]$ and he called this estimated score test. For an example, let us consider again testing for serial independence in linear regression model. For simplicity, we assume there is only one regressor X. In (3.3), we noted that the standard RS test utilizes the quantity $\sum_i \tilde{u}_i\tilde{u}_{i-1}$. However,

$$E\left[\sum_i \tilde{u}_i\tilde{u}_{i-1}\right] = -\sigma^2 \frac{\sum_i x_i x_{i-1}}{\sum_i x_i^2}$$

where $V(\varepsilon_i) = \sigma^2$. Therefore, the estimated score test will be based on

$$\sum_i \tilde{u}_i\tilde{u}_{i-1} + \tilde{\sigma}^2 \frac{\sum_i x_i x_{i-1}}{\sum_i x_i^2}$$

Monte Carlo results reported in Conniffe (1990b) show that estimated score test has better finite sample properties. It appears that some further investigation on this test will be useful in econometrics.

There is another approach to get the size of RS test close to its preassigned value in finite sample. In this case, the adjustment is done not to the score but to the test statistic. Suppose under $H_0 : \theta = \theta_0$, $E[RS] = p[1 + a(\theta_0) + O(1/n^2)]$. Then the size corrected RS test will be

$$RS^* = RS/[1 + a(\theta_0)]$$

Bartlett (1937) was probably the first to discuss an adjustment for testing homogeneity of variance in the context of LR statistic. For RS test, Harris (1985) suggested an adjustment based on Edgeworth-type expansion. Honda (1989) applied Harris' technique to RS for homoskedasticity. Dean and Lawless (1989) suggested somewhat different adjusted score tests for Poisson model. Monte Carlo results of Gurmu (1991) indicate that Dean and Lawless (1989) size correction procedure is effective in improving the small sample accuracy of RS test. Here we should note that these modifications have been done only on the basis of first moment, and it is not clear whether they would improve the approximation to the upper tail [see Cox (1988)].

Earlier we discussed the White (1982) version of the score test when the model is estimated by QMLE. His version of the test is valid even if the model is misspecified. This kind of robustification of the score test has been found to be very useful in econometrics. For example, the score test for homoskedasticity, as suggested by Koenker (1981) can be made robust by replacing $2\tilde{\sigma}^4$ in the test statistic by $n^{-1}\sum_i(\tilde{u}_i^2 - \tilde{\sigma}^2)^2$. Wooldridge (1990), following the approach of Davidson and MacKinnon (1985), suggested a general procedure for robustifying tests for a specific moment condition which does not depend on the specification of some higher order moments. So far we discussed score test based on likelihood or quasi-likelihood functions. However, score type test has wider applicability. The estimation could be based on a variety of techniques, MLE, method of moments, minimum chi-square and robust method etc. For example, a general estimating equation could be $S(\theta) = 0$. When $S(\theta) = \partial\ell(\theta)/\partial\theta$, we get the standard score test. For

robust estimation $S(\theta)$ could be chosen to be a bounded function [see Koenker (1982)]. Now taking $S(\theta)$ as the general score function, we can derive score type tests [for details see Basawa (1985)]. Neyman's $C(\alpha)$ test can also be viewed along this line. For testing $H_o : \theta_2 = \theta_{2o}$ in the presence of nuisance parameter θ_1, the estimating equation is

$$\frac{\partial \ell(\theta^+)}{\partial \theta_2} - I_{21}(\theta^+) I_{11}^{-1}(\theta^+) \frac{\partial \ell(\theta^+)}{\partial \theta_1} = 0$$

Robustness of a test can also be judged from a different point of view, by explicitly parametrizing the true model. Let us assume that the true model is represented by a density $g(y;\theta,\gamma)$ where $g(y;\theta,0) = f(y;\theta)$ which is our assumed model. Suppose we test $H_o : \theta = \theta_o$ in $f(y;\theta)$ then a natural question is to ask how can we interpret the results of the test of H_o when $g(y;\theta,\gamma)$ is true density. Haavelmo (1944) called $f(y;\theta)$ the priori admissible hypothesis. He also noted that we should study the properties of our test under certain alternatives not contained in our priori admissible hypothesis, since it is quite possible that a certain outside scheme is the true one having serious consequences for our inference. Davidson and MacKinnon (1987) and Saikkonen (1989) addressed this question and showed that RS test can have substantial power even when H_o is true if $\gamma \neq 0$. The power very much depends on the off-diagonal element of the information matrix, namely on

$$I_{\theta\gamma} = E\left[\frac{\partial lng(y;\theta,\gamma)}{\partial \theta} \cdot \frac{\partial lng(y;\theta,\gamma)}{\partial \gamma'} \right]$$

If $I_{\theta\gamma} = 0$, then the presence of γ in the density function asymptotically does not affect the inference on θ. It is possible to calculate the precise affect of γ, and using that Bera and Yoon (1990) suggested some adjustment to the RS test to adapt it for the parameter γ.

For the linear model, traditionally the score test has been developed under the assumption of normality, and then it is studied whether the limit distribution is invariant to a wide variety of densities for the error ε. A score test for the omitted variable z_i from the regression (3.1) is simply the test for $E(z_i\varepsilon_i) = 0$, and the test statistic is computed as n times the R^2 from the regression of $\underline{1}$ on $(x_i\tilde{\varepsilon}_i, z_i\tilde{\varepsilon}_i)$. If the error is not normal the RS test will be the test for $E(z_is_i) = 0$, where s_i is the score defined as the ratio of the derivative of the density and the density itself. The RS statistic, in this case, can be computed as nR^2 from the regression of $\underline{1}$ on $(x_i\hat{s}_i, z_i\hat{s}_i)$ where $\hat{s}_i$ is the estimated score. Note that under normality $s_i = -\sigma^{-2}\varepsilon_i$, and the above two approaches are the same. Other diagnostic tests can similarly be developed for non-normal cases. For these and estimation of s_i, see Pagan and Ullah (1986, ch. 6).

A result very often cited in the econometric literature is that asymptotically all the three tests LR, W and RS are equivalent under the null and local alternatives. This result is based on first-order asymptotic theory. Rao (1962) conjectured that using higher order asymptotics, RS test can be shown to be locally more powerful than the LR and W tests. In a series of papers Chandra and Joshi (1983), Chandra and Mukherjee (1984, 1985), Chandra and Samanta (1988) and Mukherjee

(1990) proved this conjecture for various cases using second and third order asymptotics. One interesting result is that the differences in power depend on Efron's (1975) curvature at θ_0. For exponential family statistical curvature is zero and large curvature indicates breakdown of some standard results in estimation and testing. Since in econometrics, we very often deal with non-exponential families, the above result has some serious implications. Amari (1985, p. 200) indicated that the W test might do well powerwise for distant alternatives since it is based on estimates under the alternative hypothesis. King's (1988) point optimal test can be viewed in this light, since he suggested to choose some parameter values away from the null hypothesis to construct a test with high power. Our conclusions will be the same as in Conniffe (1990a, p. 105). That is, the score test is the most powerful for alternatives close to the null, W test is best for distant alternatives while the LR test has some advantage for intermediate alternatives.

7. EPILOGUE

In a recent interview Professor Rao was asked about his favorite publications among his many books and papers [see DeGroot (1987)]. Part of Rao's reply was ". . . In 1947 . . . I introduced two general asymptotic test criteria called score tests for simple and composite hypotheses as alternative to Wald's tests. I find that my score test for composite hypotheses has become entrenched in the econometrics literature under a fancier name, the Lagrange Multiplier Test. So those are a few papers which I like and which have received some attention." Many of the techniques that are currently being used in econometric theory and practice have their origin in the statistics literature.

From that point of view, Rao's paper must be one of the most influential. To make the transition from statistical theory to applications, econometricians played a very prominent role. Of course, there are still many unsolved problems. Pagan (1990, p. 279) summarizes the current state rather succinctly.

> How to design a test is now well understood, and it seems unlikely that we will see much that is new in this area. How to judge it remains far more uncertain, and ultimately it will be our response to this challenge that will decide the utility of testing.

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